Theoretical Computer Science 412 (2011) 2296–2311

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Extending the notion of rationality of selfish agents: Second Order Nash equilibria[☆]

Vittorio Bilò [a,*], Michele Flammini [b]

[a] Dipartimento di Matematica "Ennio De Giorgi", Università del Salento, Provinciale Lecce-Arnesano, P.O. Box 193, 73100 Lecce, Italy

[b] Dipartimento di Informatica, Università di L'Aquila, Via Vetoio, Loc. Coppito, 67100 L'Aquila, Italy

ARTICLE INFO

Article history:
Received 21 September 2009
Received in revised form 9 July 2010
Accepted 6 January 2011
Communicated by X. Deng

Keywords:
Algorithmic Game Theory
Nash equilibria
Rational behavior

ABSTRACT

Motivated by the increasing interest of the Computer Science community in the study and understanding of non-cooperative systems, we present a novel model for formalizing the rational behavior of agents with a more farsighted view of the consequences of their actions. This approach yields a framework creating new equilibria, which we call Second Order equilibria, starting from a ground set of traditional ones. By applying our approach to pure Nash equilibria, we define the set of Second Order pure Nash equilibria and present their applications to the Prisoner's Dilemma game, to an instance of Braess's Paradox in the *Wardrop* model and to the *KP* model with identical machines.

1. Introduction

Central to the theory and study of multiplayer non-cooperative games is the notion of Nash equilibrium [39,40], due to its ability to model the rational behavior of selfish agents. All the agents (players) participating in a game have a set of strategies they can adopt and, for any combination of the strategies adopted by everyone, they obtain a certain payoff. A Nash equilibrium is a particular combination of strategies such that none of the players can improve her payoff by changing her strategy. It is well known that Nash equilibria fail in optimizing the overall satisfaction of the players in several games, the pragmatic example being the Prisoner's Dilemma. One of the reasons for this suboptimality is that players always perform deviations from a particular strategy only motivated by a transient improvement on their payoffs, without considering what will be their final payoffs when the game eventually reaches a Nash equilibrium.

This observation naturally yields the question whether an agent taking decisions only based on what will be their short term consequences without considering what these decisions will cause tomorrow can be really considered a rational one. A partial answer to this question has been given in Game Theory's literature through the definition of repeated games. The general idea repeated games are based on is that the real life is neither a one-shot game nor a disconnected set of such games, but it is indeed a bigger game in which what a player does early can influence what others choose to do later on; in other words, repetition is able to embed in the game the notion of reputation of a player. When engaged in a repeated situation, players must consider not only their short term gains but also their long term payoffs since other players may be able to deter them from exploiting their short term advantage by threatening punishment reducing their long term payoffs. Such a view point, however, can usually give benefits only when the game is repeated an arbitrarily number of times, unknown to the players. Moreover, it cannot be directly applied to a simple one-shot game.

☆ Work partially supported by the PRIN 2008 research project COGENT (COmputational and GamE-theoretic aspects of uncoordinated NeTworks), funded by the Italian Ministry of University and Research.

* Corresponding author. Tel.: +39 0832297417.
E-mail addresses: vittorio.bilo@unisalento.it, vittorio.bilo@unile.it (V. Bilò), flammini@di.univaq.it (M. Flammini).

0304-3975/$ – see front matter
doi:10.1016/j.tcs.2011.01.008

Some efforts towards the characterization of equilibria for farsighted players were produced in the 80s through the definitions of non-myopic equilibria [12] and extended non-myopic equilibria [29]. However, as we will discuss later, these approaches suffer of serious limitations.

In this paper we propose a novel model for formalizing the rational behavior of agents with a more farsighted view of the consequences of their deviations. Such an extended rationality, which can be used in any one-shot game, defines a framework yielding a new notion of equilibria which we call Second Order equilibria. They can be based upon different traditional non-cooperative equilibria. Throughout the paper we will deal with Second Order pure Nash equilibria leaving the discussion of other extensions to future works. These equilibria share some similarities with the notion of purely non-cooperative farsighted stable sets, an interesting concept, recently presented in [38] and based on the notion of stable set introduced by von Neumann and Morgenstern in [48], for modeling farsighted behavior in the n-player Prisoner's Dilemma game.

Games, equilibria and optimality

A strategic game $\mathcal{G} = (P, S_{i\in P}, \omega_{i\in P})$ is defined as follows. There is a set P of n players. Any player $p_i \in P$ has a set of strategies S_i and the set $S = S_1 \times \cdots \times S_n$ is called the set of all possible states of the game. The payoff function $\omega_i : S \to \Re$ defines the cost[1] that player p_i has to incur when the game is on state $s \in S$. Usually, each game has an associated global function $\gamma : S \to \Re$, called the social function, that is required to be optimized.

Let $s = (s_1, \ldots, s_i, \ldots, s_n)$ and $s' = (s_1, \ldots, s'_i, \ldots, s_n)$ be two states of $\mathcal{G}$ such that $\omega_i(s') < \omega_i(s)$. We call the transition of game $\mathcal{G}$ from s to s' an improving step performed by player p_i. A pure Nash equilibrium is a state in which no player possesses an improving step. Unfortunately, pure Nash equilibria are not guaranteed to exist for all games, thus Nash himself generalized this definition by introducing the concept of mixed strategy. A mixed strategy for player p_i is a probability distribution on the set of strategies S_i. The payoff obtained by p_i is now defined as the expected value of the related random variable and the definition of mixed Nash equilibrium is obtained consequently. The property of mixed Nash equilibria, stated by Nash's famous Theorem, is that they exist for any finite game. However, there are cases in which the use of mixed strategies is unrealistic or unacceptable. Throughout the paper we will only deal with pure Nash equilibria and will refer to them simply as Nash equilibria.

The evolution of a game $\mathcal{G}$ resulting from the interactions among players performing improving steps can be easily captured by a graph $G_{\mathcal{G}} = (N, A)$, called the state graph of $\mathcal{G}$, where $N = S$ and A is such that there exists an edge between s and s' if and only if there exists an improving step from s to s'. An important issue related to the notion of Nash equilibrium is that of convergence towards such an equilibrium point, also called finite improvement path property. A game $\mathcal{G}$ is said to be convergent if for any $s \in S$, any sequence of improving steps starting from s ends in a Nash equilibrium, or, analogously, $\mathcal{G}$ does not admit an infinite sequence of improving steps. By using the representation of $\mathcal{G}$ through its state graph, we have that $\mathcal{G}$ is convergent if and only if $G_{\mathcal{G}}$ is acyclic. The notion of state graph and convergence is of fundamental importance in those games arising in highly dynamic systems, where players frequently enter and leave the game thus keeping it in a continuous phase of evolution towards a Nash equilibrium. As we will see later on, the state graph, with its ability to capture the game's dynamic, plays a crucial role in our model.

Two metrics have been introduced in the literature in order to capture the loss of optimality yielded by non-cooperative equilibria, that is the price of anarchy [41] and the price of stability [2]. Let $\mathcal{N}(\mathcal{G})$ be the set of equilibria of game $\mathcal{G}$ and s^* be a state optimizing the social function γ. The price of anarchy of $\mathcal{G}$ is defined as $PoA(\mathcal{G}) = \max_{s\in\mathcal{N}(\mathcal{G})} \frac{\gamma(s)}{\gamma(s^*)}$, while the price of stability of $\mathcal{G}$ is defined as $PoS(\mathcal{G}) = \min_{s\in\mathcal{N}(\mathcal{G})} \frac{\gamma(s)}{\gamma(s^*)}$.

Repeated games

The basis of repeated games were posed in [4,34,46]. Consider a game $\mathcal{G}$, which we will call the stage game. The repeated game obtained from $\mathcal{G}$ is defined in the following way. The stage game is played at each discrete time period $t = 0, 1, 2, \ldots, T$ and at the end of each period, all players observe the realized states. The game is finitely repeated if $T < \infty$ and is infinitely repeated otherwise. Let $s^t = (s^t_1, \ldots, s^t_n)$ be the state realized at period t (and so s^t_i is the strategy played by player p_i in that period), and denote the initial history by h^0. A history of the repeated game in time period $t \geq 1$ is denoted by h^t and is simply the sequence of the realized states from all periods before t, that is, $h^t = (s^0, s^1, \ldots, s^{t-1})$. Let $H^t = (S)^t$ be the space of all possible period-t histories. After any nonterminal history, all players $p_i \in P$ simultaneously choose a strategy $s_i \in S_i$. Because every player observes h^t, a pure strategy for player p_i in the repeated game is a sequence of functions, $f_i(h^t) : H^t \to S_i$, that assign possible period-t histories $h^t \in H^t$ to strategies $s_i \in S_i$. That is, $f_i(h^t)$ denotes a strategy s_i for player p_i after history h^t. So, a strategy for player p_i in the repeated game becomes $f_i = (f_i(h^0), f_i(h^1), \ldots, f_i(h^T))$ where it may well be the case that $T = \infty$. We now define the players' payoff functions for infinitely repeated games (for finitely repeated games, the payoffs are usually taken to be the time average of the per-period payoffs). Since the only terminal histories are the infinite ones and because each period's payoff is the payoff from the stage game, we must describe how players evaluate infinite streams of payoffs of the form $(\omega_i(s^0), \omega_i(s_1), \ldots)$. There are several alternative specifications in the literature but we shall focus on the case where players discount future utilities using a discount factor $\delta \in (0, 1)$. Player p_i's payoff for the infinite sequence $(s_0, s_1, \ldots)$ is given by the discounted sum of per-period payoffs: $u_i = (1-\delta)\sum_{t=0}^{\infty} \delta^t \omega_i(s^t)$.

1 In the case in which the payoffs represent a benefit for the players, one can always be consistent with this definition by changing the signs of the values.

Network games and selfish routing

In recent years a considerable research effort has been devoted to the estimation of the price of anarchy of different network games. The reasons for such an interest come from the affirmation of the Internet and, in general, of huge unregulated networks, where the traffic generated by the users is not controlled by some central authority, but it is rather the outcome resulting from the interaction of the users when routing their traffic selfishly and independently on the network. Two major models have been deeply investigated by researchers: the *KP* model [32] and the *Wardrop* model [17,49], both being convergent games.

In the *KP* model there are n players and m parallel links. Each player owns a certain unsplittable traffic and wants to route it through one of the links. This game can also be interpreted as the non-cooperative version of the problem of scheduling n jobs on m parallel machines. The payoff obtained by a player is the completion time of the chosen machine. The social function is the makespan, that is the maximum completion time of all the machines. This model has been extensively studied in [15,16,18,20–22,25,31,32,36,37].

In the *Wardrop* model there are infinitely many players who want to route their traffic over an arbitrary network. There is a convex latency function associated with each link which is defined in terms of its load. The traffic can be split into arbitrary pieces each being handled by a selfish player, so that unregulated traffic can be modelled as a network flow. The payoff obtained by a player is the sum of the latencies experienced on the edges she uses. The social function is the sum of the products between the payoff of each player and the amount of traffic she owns. The *Wardrop* model can also be seen as a congestion game [42] where all the players own the same infinitesimal small amount of traffic. This model was defined in [49] and then studied in [6,7,11,17]. Recent papers devoted to the study of its price of anarchy include [43–45].

Some other works, which have dealt with the study of the price of anarchy of pure and/or mixed Nash equilibria in different network games, can be found in [3,8–10,19,23,24,35].

Our contribution

Critics and improvements upon the classical notion of pure Nash equilibria have had different targets such as *existence*, as in the case of correlated and sink equilibria, *stability*, as in the case of stable equilibria and stochastic adjustment models, *irrationality*, as in the case of perfect and sequential equilibria, and *need of complete information*, as in the case of Bayesian equilibria.

Our model of rationality defines agents taking into account the long term effects of their choices even during a one-shot game. An agent knows that she is part of a multiplayer game and also knows that the game will not stop right after she has performed an improving step. Assume that in state s player p_i possesses an improving step and that, if she performs such a move, a sequence of improvements begins leading the game toward a Nash equilibrium s'. In this scenario, p_i is mostly interested in comparing the payoff she is experiencing in s with the one she can get at s'. The idea is that, if $\omega_i(s')$ is worse than $\omega_i(s)$, p_i is damaged by the consequences of her improving step, hence she has better not to perform it. When more than just one equilibrium can be reached from a particular state, by following a classical worst-case analysis, we assume that the agent will compare the current state with the equilibrium yielding the worst payoff for her. Such a view point is clearly based upon the definition of a ground set of equilibria the agents will compare a generic state with. According to these comparisons, if an agent detects a dangerous improving step, she will never perform it, hence the state graph of the game will be pruned by removing all such improving steps. Informally speaking, since the degree of the graph decreases, it may be the case that new sinks in the graph emerge, thus generating new equilibria. This process may be iterated recursively until a fixed point is reached and the final set of desired equilibria is created. We call such a set the set of Second Order equilibria. Using different definitions of equilibrium for defining the ground set, we can achieve different sets of Second Order equilibria. In this paper we concentrate our attention on the definition and the evaluation of Second Order Nash equilibria, that is with a ground set given by the set of Nash equilibria. In particular, we present applications of these equilibria to the Prisoner's Dilemma, to an instance of Braess's Paradox [11] in the *Wardrop* model and to the *KP* model with identical machines.

Related works

Other equilibrium concepts have been introduced and studied in the literature. We recall here perfect equilibria [47], sequential equilibria [33], stable equilibria [30], stochastic adjustment models [28], correlated equilibria [5], Bayesian equilibria [27] and, recently, sink equilibria [26]. In particular, correlated and sink equilibria are generalizations of mixed and pure Nash equilibria respectively, while the others are refinements of mixed (and consequently also pure) Nash equilibria.

Farsighted players were first introduced in [12] for 2-player games in which each player has exactly two strategies, i.e., the most basic case of non-cooperative games. It is postulated that, when a player (let us say the row player) contemplates a departure from a state s, she considers first her own move, then the column player's response, then her counter-response and so on, in a strictly alternating sequence. Such a subgame is called the departure game for the row player starting from s. If the departure game ends at a state whose payoff for the row player is not greater than her payoff in s, s is declared to be non-myopically stable for the row player. A profile is a non-myopic equilibrium if it is non-myopically stable for both the row and the column player. However, no specific rules are given in order to determine the final state of a departure game, except for the case in which the player who is asked to move is already at one of her most preferred states (i.e., states giving her the best possible payoff). In all the other cases in which this situation does not happen, the initial state s from which the departure game generates is not eligible to be declared non-myopically stable. This idea was extended in [29], where backward induction was used to obtain an non-empty set of states representing the solution of each departure game. A state s is declared to be well determined for the row player if all arbitrarily long departure games starting from s have the same unique final outcome $F(s)$. The state s is extended non-myopically stable for the row player if it is well determined for such a player and $F(s) = s$. Again, s is an extended non-myopic equilibrium if it is extended non-myopically stable for both the row and the column player. However, there might be cases in which none of the states of the game is well determined for both players, as well as cases in which for each state s which is well determined for both players, it holds $F(s) \neq s$. Finally [1] extended the approach in [29] to the case of repeated games by showing examples of extended farsighted behavior applied to the Prisoner's Dilemma and to the Matching Pennies games, but without developing a formal general theory for this model.

A more interesting and general approach can be found in [38], in which the notion of purely non-cooperative farsighted stable set is presented as a solution concept for the n-player Prisoner's Dilemma. To this aim, a state y is said to indirectly dominate a state x if there exists a sequence of states starting with x and ending with y such that each state is obtained from the previous one after the deviation (not necessarily an improving one) of a player and the payoff each deviating player gets in the previous state is higher than the payoff she gets in y. Thus, each deviation is an improving one when compared with y, i.e., the final state of the sequence. A subset of states K is called a purely non-cooperative farsighted stable set if, for each pair of states $x, y \in K$, neither x indirectly dominates y nor y indirectly dominates x and for each state $x \notin K$, there exists $y \in K$ such that y indirectly dominates x.

Finally, farsighted equilibria are defined and analyzed in [13] for keyword auctions. In such a setting, because of the particular nature of the game, the classical notion of pure Nash equilibrium reveals to be not completely satisfactory. In fact, once fixed the strategies adopted by the other players, a given player usually has an infinite number of best responses, i.e. possible bids falling in a continuous range of values. Because of this, the player can look ahead and choose, among all the possible equivalent bids, the one which will possibly trigger a sequence of reactions by the other players leading to a final gain for her without risking to get a lower utility.

Comparison with previous works and significance

What makes our approach different from the one yielded by repeated games is that the awareness of a long term disadvantage already emerges in a one-shot game, without the need of resorting to repetition. In this latter case, in fact, disadvantages may occur for a player as a consequence of the bad reputation she made of herself in early stages of the game. Moreover, a change in the definition of available strategies and payoff functions needs to be introduced in order to comply with the definitions characterizing repeated games. In our model, on the contrary, a player looks to the disadvantages which may occur even during the evolution of a one-shot game as a consequence of the sequence of improving steps she can create with her first defection.

The notion of non-myopic equilibrium, as well as its extensions, heavily relies on the simple structure of the analyzed games. In fact, generalizations of these concepts to n-player games with arbitrary number of strategies per player seem to be not easily achievable. Moreover, even in their basic cases of definitions, these farsighted equilibria are not always completely characterized. Our Second Order pure Nash equilibria, instead, are well defined and do always exist in any convergent game.

Purely non-cooperative farsighted stable sets share some ideas with our notion of Second Order pure Nash equilibria, however, they produce incomparable outcomes, since it is possible to provide examples of Second Order Nash equilibria not belonging to the purely non-cooperative farsighted stable set as well as states belonging to the purely non-cooperative farsighted stable set which are not Second Order Nash equilibria. The main reason for such a difference is caused by the fact that we rely our notion of dominance on sequences of improving deviations, i.e., the players continue to act in a purely myopic selfish manner: their farsightedness just refrain them from following illusory improving deviations. In the model of [38], instead, players are even willing to suffer a loss in the brief term if this can give them an improvement in the long term. Both these approaches are clearly reasonable; we argue that ours is more appropriate when considering an evolutionary game which can be stopped at any time, or after the players have performed a certain number of deviations, and these information about the game's duration are unknown to the players.

Paper organization

In the next section we give the formal definition of Second Order Nash equilibria. In Section 3 we illustrate how they apply to the Prisoner's Dilemma game and also give a brief comparison with the approach of repeated games. In Section 4 we discuss applications of our Second Order Nash equilibria to the two main models for selfish routing, that is, the *Wardrop* model and the *KP* model. In Section 5 we introduce other possible models of farsighted selfish agents and finally, in the last section we give conclusive remarks and open questions.

2. Second Order Nash equilibria

We define our notion of farsightedness in the following way. Consider an integer $k \geq 0$, a state s and a player i who can perform an improving step in s leading the game to a new state s'. If there exists a pure Nash equilibrium s'' which can be reached from s' after k improving steps, and such that $\omega_i(s) < \omega_i(s'')$, then player i has no convenience in leading the game from s to s'. If these conditions hold for all possible states s' which can be reached from s after an improving step performed by player i, then we say that s is stable for player i with respect to pure Nash equilibria when considering a horizon of k improving steps. If such a stability holds for each player i who can perform an improving step in s, we say that s is stable with respect to pure Nash equilibria when considering a horizon of k improving steps. But now we have new equilibria other than pure Nash ones in our game and so we can no longer restrict ourselves to pure Nash equilibria when comparing the payoffs of a given state s with those of all the equilibria which can be reached from s' after player i has performed an improving step leading the game from s to s'. The correct notion of stability, in fact, can only be achieved by making our definition recursive, i.e., by using the set of equilibria we are currently defining as the set of stable states to compare the payoffs of a given state with.

To this aim, we introduce the following definitions and notation. We first propose a generalization of the state graph related to a game $\mathcal{G}$. Given a set of equilibrium states $E \subseteq S$, let $G_{\mathcal{G},E} = (N, A)$ be the directed graph in which $N = S$ and A is such that there exists an edge between s and s' if and only if there exists an improving step from s to s' and $s \notin E$. Clearly, $G_{\mathcal{G},\emptyset}$ coincides with the state graph $G_{\mathcal{G}}$. If p_i is the unique player changing her strategy from s to s', we label the arc $\langle s, s'\rangle$ with the index i.

We define $\rho_E(s)_i^k$ as the set of all the states of $\mathcal{G}$ that can be reached starting from s by following a path of length at most k whose first arc is labelled with index i in the graph $G_{\mathcal{G},E}$. The set $\rho_E(s)^k = \bigcup_{i=1}^n \rho_E(s)_i^k$ will denote the set of all the states that can be reached from s by following a path of length at most k in the graph $G_{\mathcal{G},E}$. When $E = \emptyset$, we will simply remove the subscript E from the notation. We also define $P(s)$ as the set of players that can perform an improving step starting from state s.

We now give a recursive definition of the new set of equilibria that will be further clarified in the following.

Definition 1. Let $\mathcal{G}$ be a convergent game. The set $N^k(\mathcal{G}) = \{s \in S : \forall p_i \in P(s)$ and $\forall s' \in \rho(s)_i^1, \exists s'' \in N^k(\mathcal{G})$ such that $s'' \in \rho_{N^k(\mathcal{G})}(s')^k$ and $\omega_i(s) < \omega_i(s'')\}$ is the set of all the Second Order k-Nash equilibria of game $\mathcal{G}$, for any integer $k \geq 0$.

Intuitively, this rather involved definition says that a state s is a Second Order k-Nash equilibrium, for some integer $k \geq 0$, if all the players who can perform an improving step in s would experience, in one of the Second Order k-Nash equilibria resulting from an evolutive process of at most k improving steps taking place after their first defection, a payoff which is worse than the one they get in state s. Such a definition is clearly recursive. However, in the following we show that it is well posed, in the sense that it admits a unique set of solutions or a fixed point. First of all, we prove that $N^0(\mathcal{G})$ coincides with the set of the Nash equilibria of $\mathcal{G}$ and that each Nash equilibrium is a Second Order k-Nash equilibrium, for any integer $k \geq 1$.

Lemma 1. *$N^0(\mathcal{G}) = \{s \in S : s$ is a Nash equilibrium$\}$ and $N^0(\mathcal{G}) \subseteq N^k(\mathcal{G})$, for any integer $k \geq 1$.*

Proof. Let s be a Nash equilibrium. Since $P(s) = \emptyset$, we have that s trivially belongs to $N^k(\mathcal{G})$ for any integer $k \geq 0$. Now consider a state $s \in N^0(\mathcal{G})$ and suppose that s is not a Nash equilibrium. Consider a player $p_i \in P(s)$ and a state $s' \in \rho(s)_i^1$. Since $k = 0$, we have that $\rho_{N^k(\mathcal{G})}(s')^k = \{s'\}$, thus, in order for s to verify Definition 1, it must be $s' \in N^0(\mathcal{G})$ and $\omega_i(s) < \omega_i(s')$. But this last inequality contradicts the fact that $s' \in \rho(s)_i^1$, thus $N^0(\mathcal{G})$ contains only Nash equilibria. □

We now define an algorithm constructing a set of states $\tilde{N}^k(\mathcal{G})$ that we will after show to coincide with $N^k(\mathcal{G})$. To this aim, we first introduce some necessary notation. Given a directed graph $G = (N, A)$ and a set of vertices $T \subseteq N$, we define *leaves*(T) as the maximal subset of T such that for any $s \in$ *leaves*(T) and $s' \in T$ there exists no (s, s')-directed path in G.

Lemma 2. *For any acyclic directed graph $G = (N, A)$ and any non-empty subset of vertices $T \subseteq N$, leaves(T) is unique and leaves$(T) \neq \emptyset$.*

Proof. Since G is acyclic, there exists a lexicographic ordering on its node set N. This implies that *leaves*$(T) \neq \emptyset$. Now consider two sets $L =$ *leaves*(T) and $L' =$ *leaves*(T) such that $L \neq L'$. Without loss of generality, we can assume that there exists a vertex $s \in L \setminus L'$. By the definition of *leaves*(T), there exists no (s, s')-directed path in G for any $s' \in T$. This means that $L' \cup \{s\} \subseteq$ *leaves*(T) thus contradicting the maximality of L'. □

As a consequence of the above lemma, the following corollary immediately follows.

Corollary 1. *For any acyclic directed graph $G = (N, A)$, leaves$(T) = \emptyset$ if and only if $T = \emptyset$.*

The algorithm Construct $\tilde{N}^k(\mathcal{G})$ for determining $\tilde{N}^k(\mathcal{G})$ is defined as follows.

Construct $\tilde{N}^k(\mathcal{G})$**:**

1. $\tilde{N}^k(\mathcal{G}) \leftarrow N^0(\mathcal{G})$
2. $T(k) \leftarrow \{s \in S \setminus \tilde{N}^k(\mathcal{G}) : \forall p_i \in P(s)$ and $\forall s' \in \rho(s)_i^1, \exists s'' \in \tilde{N}^k(\mathcal{G})$ such that $s'' \in \rho_{\tilde{N}^k(\mathcal{G})}(s')^k$ and $\omega_i(s) < \omega_i(s'')\}$
3. **if** $T(k) \neq \emptyset$
   (a) $\tilde{N}^k(\mathcal{G}) \leftarrow \tilde{N}^k(\mathcal{G}) \cup$ *leaves*$(T(k))$
   (b) **goto** 2
4. **else return** $\tilde{N}^k(\mathcal{G})$

Clearly, since at each step $\tilde{N}^k(\mathcal{G})$ grows, the algorithm terminates. Moreover, because of Lemma 2, we have that $\tilde{N}^k(\mathcal{G})$ is unique. In the following theorem we show that $N^k(\mathcal{G}) = \tilde{N}^k(\mathcal{G})$ for any integer $k \geq 0$, thus proving the uniqueness of $N^k(\mathcal{G})$.

Theorem 1. *Let $\mathcal{G}$ be a convergent game then $N^k(\mathcal{G}) = \tilde{N}^k(\mathcal{G})$ for any integer $k \geq 0$.*

Proof. The case $k = 0$ can be easily shown by noting that $\tilde{N}^0(\mathcal{G}) = \{s \in S : s \text{ is a Nash equilibrium}\}$. In order to prove the general claim consider the state graph $G_{\mathcal{G}}$. Since $\mathcal{G}$ is convergent, $G_{\mathcal{G}}$ is acyclic. For any $\tilde{N}^k(\mathcal{G})$ satisfying Definition 1, let $T = N^k(\mathcal{G}) \cup \tilde{N}^k(\mathcal{G}) \setminus N^k(\mathcal{G}) \cap \tilde{N}^k(\mathcal{G})$ and consider a state $s \in leaves(T)$ in $G_{\mathcal{G}}$. Because of Lemma 1 and line 1 of Construct $\tilde{N}^k(\mathcal{G})$, s cannot be a Nash equilibrium. We have to distinguish between two cases:

- $s \in N^k(\mathcal{G}) \setminus \tilde{N}^k(\mathcal{G})$. This means that
  1. $\forall p_i \in P(s)$ and $\forall s' \in \rho(s)_i^1$, $\exists s'' \in N^k(\mathcal{G})$ such that $s'' \in \rho_{N^k(\mathcal{G})}(s')^k$ and $\omega_i(s) < \omega_i(s'')$ and,
  2. $\exists p_i \in P(s)$ and $\exists s' \in \rho(s)_i^1$ such that $\forall s'' \in \tilde{N}^k(\mathcal{G})$ such that $s'' \in \rho_{\tilde{N}^k(\mathcal{G})}(s')^k$ it holds $\omega_i(s) \geq \omega_i(s'')$.

  Consider the pair (p_i, s') verifying condition 2 and a state s'' verifying condition 1 for the pair (p_i, s'). If $s'' \in \tilde{N}^k(\mathcal{G})$, since s'' violates condition 2, there must be $s'' \in \rho_{N^k(\mathcal{G})}(s')^k$ and $s'' \notin \rho_{\tilde{N}^k(\mathcal{G})}(s')^k$. This implies the existence of at least a state $\bar{s} \in \tilde{N}^k(\mathcal{G})$ belonging to a directed path form s' to s'' in $G_{\mathcal{G}}$ such that $\bar{s} \notin N^k(\mathcal{G})$. Such an $\bar{s}$ belongs to T and there exists an $(s, \bar{s})$ directed path in $G_{\mathcal{G}}$. Thus s cannot belong to $leaves(T)$. If $s'' \notin \tilde{N}^k(\mathcal{G})$, then $s'' \in T$ and there exists an (s, s'') directed path in $G_{\mathcal{G}}$. Thus s again cannot belong to $leaves(T)$.
- $s \in \tilde{N}^k(\mathcal{G}) \setminus N^k(\mathcal{G})$. This case can be proved symmetrically by exchanging the role of $N^k(\mathcal{G})$ and $\tilde{N}^k(\mathcal{G})$ in the previous one.

We have shown $leaves(T) = \emptyset$. As a consequence of Corollary 1 we have that $T = \emptyset$ and this holds if and only if $N^k(\mathcal{G}) = \tilde{N}^k(\mathcal{G})$, hence the claim. □

In the following lemma we show that there exists a value k^* for which all the sets $N^k(\mathcal{G})$ become the same for any $k \geq k^*$.

Lemma 3. *Let $\mathcal{G}$ be a convergent game and let k^* be the diameter of $G_{\mathcal{G}}$ minus 1, then $N^{k^*}(\mathcal{G}) = N^k(\mathcal{G})$ for any integer $k \geq k^*$.*

Proof. By exploiting Theorem 1, it suffices to show that $\tilde{N}^k(\mathcal{G}) = \tilde{N}^{k^*}(\mathcal{G})$ for any integer $k \geq k^*$. Considering algorithm Construct $\tilde{N}^k(\mathcal{G})$, this immediately follows by observing that for every integer $k \geq k^*$, $T(k) = T(k^*)$ and thus $leaves(T(k)) = leaves(T(k^*))$. □

We can now define the general notion of Second Order Nash equilibrium as follows.

Definition 2. Given a convergent game $\mathcal{G}$, each state $s \in N^{k^*}(\mathcal{G}) =_{def} N(\mathcal{G})$ is a Second Order Nash equilibrium.

As a consequence of Lemma 1 we have that the introduction of our notion of extended rationality through the definition of Second Order Nash equilibria is able to enrich the set of equilibria states of a given non-cooperative game. If one looks to the dynamic evolution of a game and, in particular, to its representation by means of the state graph, it is thus possible to appreciate that for any $s \in N(\mathcal{G}) \setminus N^0(\mathcal{G})$ all the improving steps possessed by the set of players $P(s)$ are never performed when the game is on state s. By exploiting the same relationships relating the notions of improving step and Nash equilibrium, we can define a Second Order improving step for player p_i as an improving step (s, s') such that $\forall s'' \in N(\mathcal{G})$ with $s'' \in \rho_{N(\mathcal{G})}(s')^{k^*}$ it holds $\omega_i(s) \geq \omega_i(s'')$ and then define also the notion of Second Order state graph. Clearly, we will have that a Second Order Nash equilibrium is a state admitting no Second Order improving steps or, analogously, any sink in the Second Order state graph. The importance of these definitions can be appreciated when analyzing the problem of convergence towards equilibria states. In fact, by using the extended rationality, we can think of using the Second Order state graph, rather than the traditional one, in order to model the dynamic evolution of the game during time. This could be quantitatively studied in order to understand whether extended rationality may yield faster convergence and/or convergence towards better equilibria (see, for instance the example in Section 4.2).

3. An illustrating example: the Prisoner's Dilemma

(Prisoner's Dilemma). Two suspects in a major crime are held in separate cells. There is enough evidence to convict each of them of a minor offense, but not enough evidence to convict either of them of the major crime unless one of them acts as an informer against the other (finks). If they both stay quiet, each will be convicted of the minor offense and spend one year in prison. If one and only one of them finks, he will be freed and used as a witness against the other, who will spend three years in prison. If they both fink, each will spend two years in prison.

This situation can be modelled as a strategic game in which we have two players p_1 and p_2. The set of strategies is the same for both of them and is $S_i = \{Quiet, Fink\}$, for $i \in \{1, 2\}$. Finally, the payoff function is shown in the table depicted in Fig. 1.

		Player 2: Quiet	Player 2: Fink
Player 1	Quiet	1,1	3,0
	Fink	0,3	2,2

Fig. 1. The Prisoner's Dilemma game.

As can be easily seen by inspection, the game admits one Nash equilibrium represented by the state $\{Fink, Fink\}$. This non-cooperative solution contrasts with the state $\{Quiet, Quiet\}$ in which both players enjoy a better payoff. Based on this unrealistic outcome, several critics have been risen to the notion of Nash equilibrium as a meaningful modeling instrument for rational behavior. However, if the game is repeatedly played by the same two players (Repeated Prisoner's Dilemma (RPD)), then possibilities for cooperation may emerge. For example, in the RPD game, a strategy may specify

$$f_i(h^0) = Quiet,$$
$$f_i(h^t) = \begin{cases} Quiet & if\ a^\tau = \{Quiet, Quiet\},\ for\ \tau = 0, 1, \dots, t-1, \\ Fink & otherwise. \end{cases}$$

This strategy will read: "begin by cooperating in the first period, then cooperate as long as both players have cooperated in all previous periods, defect otherwise". (This strategy is called the grim-trigger strategy because even one defection performed by anyone of the two players triggers a retaliation that lasts forever). Consider the case in which RPD is played $T < \infty$ times and let us evaluate the payoffs of both players in order to detect a Nash equilibrium. It is possible to show that every Nash equilibrium in this case generates the always *Fink* outcome. To see this, let s denote some Nash equilibrium. Both players will *Fink* in the last period T for any history h^T because doing so increases their period-T payoff and because there are no future periods in which they might be punished. Since players will always *Fink* in the last period along the equilibrium path, if player p_i conforms to her equilibrium strategy in period $T-1$, her opponent will *Fink* at time T, and therefore player p_i has no incentive not to *Fink* in $T-1$. An induction argument completes the proof. This already shows that modeling farsighted players through finite repeated games cannot yield cooperation among the players.

The set of equilibria of an infinitely repeated game can be very different from that of the corresponding finitely repeated game because players can use self-enforcing rewards and punishments that do not unravel from the terminal date. In particular, because there is no fixed last period of the game, in which both players will surely *Fink*, in the RPD game players may be able to sustain cooperation by the threat of "punishment" in case of defection. While in the case of finitely repeated games a strategy can explicitly state what to do in each of the T periods, specifying strategies for infinitely repeated games is trickier because it must specify actions after all possible histories, and there is an infinite number of these. To calculate the payoffs, we need to specify the strategies for both players. For example, it is quite easy to see that the always *Fink* strategy remains a Nash equilibrium for both players. However, let us check a more significant case, that is, whether the state in which both players adopt the grim-trigger strategy is a Nash equilibrium. If both players follow this strategy, the outcome will be cooperation in each period whose average discounted value is $(1-\delta)\sum_{t=0}^{\infty}\delta^t\omega_i(\{Quiet, Quiet\}) = (1-\delta)\sum_{t=0}^{\infty}\delta^t = 1$. Consider the best possible deviation for p_1. For such a deviation to be profitable, it must produce a sequence of states which has at least a defection by some player in some period. If p_2 follows grim-trigger, she will not defect until p_1 defects, which implies that a profitable deviation must involve a defection by p_1. Let τ be the first period in which p_1 defects. Since p_2 follows grim-trigger, she will play *Fink* from period $\tau+1$ onward. Therefore, the best deviation for p_1 generates the following sequence of states:

$$\underbrace{\{Quiet, Quiet\}, \dots, \{Quiet, Quiet\}}_{\tau\ times}, \{Fink, Quiet\}, \{Fink, Fink\}, \{Fink, Fink\}, \dots$$

which generates the following sequence of payoffs for p_1:

$$\underbrace{1, \dots, 1}_{\tau\ times}, 0, 2, 2, \dots$$

The average discounted value of this sequence is

$$\begin{aligned}(1-\delta)[1+\delta+\delta^2+\cdots+\delta^{\tau-1}+2\delta^{\tau+1}+2\delta^{\tau+2}+\cdots] &= (1-\delta)\left[\sum_{t=0}^{\tau-1}\delta^t + \sum_{t=\tau+1}^{\infty}2\delta^t\right] \\ &= (1-\delta)\left[\frac{1-\delta^\tau}{1-\delta}+\frac{2\delta^{\tau+1}}{1-\delta}\right] \\ &= 1-\delta^\tau+2\delta^{\tau+1}.\end{aligned}$$

Solving the following inequality for δ yields the discount factor necessary to sustain cooperation:

$$1-\delta^\tau+2\delta^{\tau+1} \geq 1.$$

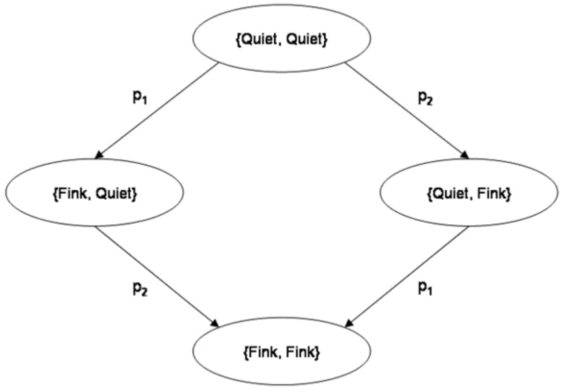


Fig. 2. The state graph of the Prisoner's Dilemma.

That is, for any $\delta \geq \frac{1}{2}$, i.e., if the players are patient enough, then the outcome of the situation in which both players adopt the grim-trigger strategy, yielding cooperation in all periods, is a Nash equilibrium for the infinitely RPD.

On the other hand, it can be easily shown that $N(\mathcal{G}) = \{\{Fink, Fink\}, \{Quiet, Quiet\}\}$. In fact we have $\{Fink, Fink\} \in N(\mathcal{G})$ as $\{Fink, Fink\} \in N^0(\mathcal{G})$ and $\{Quiet, Quiet\} \in N(\mathcal{G})$ since, as can be seen in Fig. 2, $\{Fink, Fink\} \in \rho(\{Quiet, Quiet\})_i^1$ for $i \in \{1, 2\}$ and $\omega_i(\{Fink, Fink\}) > \omega_i(\{Quiet, Quiet\})$ for $i \in \{1, 2\}$ and no other state belongs to $N(\mathcal{G})$.

This shows that our notion of farsighted behavior modelled through Second Order equilibria can yield the same outcome of the model of infinitely repeated games and has the advantage of not changing the nature of the game which remains uneffected in the set of strategies as well as in the payoff functions of the players.

4. Some applications of Second Order Nash equilibria

The power of Second Order equilibria lies in the fact that they introduce a sort of cooperation among the players in the following particular sense: the players are interested in not hurting each other, that is, in not leading the game to a state that is worse than the current one for each of them. By quoting a classical Italian proverb we can say that they *"do not awake the sleeping dog"*.

Quantitatively speaking, the effects of the introduction of Second Order Nash equilibria are the following.

Proposition 1. *In any game $\mathcal{G}$, the price of stability of Second Order k-Nash equilibria is not worse than that of Nash equilibria, while the price of anarchy of Nash equilibria is not worse than that of Second Order k-Nash equilibria, for any $k \geq 1$.*

Proof. Since $N^0(\mathcal{G}) \subseteq N^k(\mathcal{G})$ for any $k \geq 1$, the claim holds trivially. $\square$

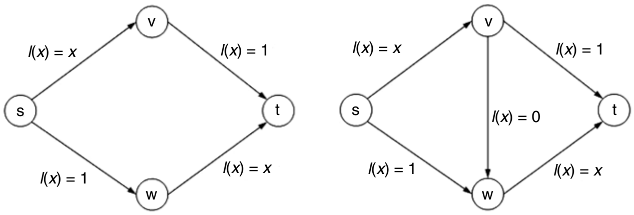
Proposition 2. *In any game $\mathcal{G}$, the maximum number of steps needed to reach a Second Order k-Nash equilibrium is not greater than the maximum number of steps needed to reach a Nash equilibrium, while the minimum number of steps needed to reach a Nash equilibrium is not greater than the minimum number of steps needed to reach a Second Order k-Nash equilibrium.*

Proof. The claim follows from the fact that the set of edges (and, consequently, the set of paths) in the Second Order state graph of $\mathcal{G}$ is a subset of the set of edges in the state graph of $\mathcal{G}$. $\square$

This means that, if we agree on the fact that "in the real world" selfish players possess the extended rationality characterizing our notion of Second Order Nash equilibria, worst-case convergence towards an equilibrium state may be better, while best-case convergence may be worse; on the contrary best-case outcomes are expected to be even better, while worst-case ones are expected to be even worse than the ones obtained through the use of Nash equilibria. In particular, as we have seen in the previous section, a good behavior of Second Order Nash equilibria can be appreciated when applied to the Prisoner's Dilemma. We show in what follows that another well-known paradox in Game Theory, namely, Braess's paradox in the *Wardrop* model can be avoided with the help of our extended rationality.

4.1. Braess's paradox

Suppose one unit of traffic needs to be routed from s to t in the first network of Fig. 3, where each edge is labelled with its latency function of the link congestion x. In the unique flow at Nash equilibrium, which coincides with the optimal flow, half of the traffic takes the upper path and the other half travels along the lower path, and thus all agents are routed on a path of latency $\frac{3}{2}$. Next suppose a fifth edge of latency 0 (independent of the congestion) is added to the network, with the result shown in Fig. 3(b). The optimal flow is unaffected by this augmentation (there is no way to use the new link to decrease the total latency) while in the new (unique) flow at Nash equilibrium all traffic follows the path $s \to v \to w \to t$; here, the latency experienced by each agent is 2. Thus, the intuitively helpful (or at least innocuous) action of adding a new zero-latency link may negatively impact on the payoffs of all of the agents.



(a) Before. (b) After.

Fig. 3. Braess's Paradox.

Let Π_1 be the path $s \to v \to t$, Π_2 be the path $s \to w \to t$, Π_3 be the path $s \to v \to w \to t$ and define f_i as the amount of flow routed on path Π_i. We can thus denote a state $s \in S$ as $s = (f_1, f_2, f_3)$ by specifying the amount of flow routed on the three different paths.

Since our model of Second Order equilibria, based on the structure of the state graph, is clearly a discrete one, we consider the instance of Braess's Paradox in which the unitary flow representing the traffic on the network is split into infinitely many atomic pieces all having the same infinitesimal dimension $\epsilon > 0$ and denote this game by $\mathcal{G}_\epsilon$. This assumption does not change the properties of the *Wardrop* model as well as those of Braess's Paradox instance $\mathcal{G}$ since it is easy to see that $\mathcal{G} = \lim_{\epsilon\to 0} \mathcal{G}_\epsilon$. In particular, the set of Nash equilibria becomes $N^0(\mathcal{G}_\epsilon) = \{(0, 0, 1), (\epsilon, 0, 1-\epsilon), (0, \epsilon, 1-\epsilon), (\epsilon, \epsilon, 1-2\epsilon)\}$ which collapses to $N^0(\mathcal{G}) = \{(0, 0, 1)\}$ when ϵ tends to zero.

The main result of this section is the characterization of the set of Second Order Nash equilibria for $\mathcal{G}_\epsilon$.

The following lemma will be widely used inside the proof of Theorem 2.

Lemma 4. *Given two states $s = (f_1, f_2, f_3)$ and $s'' = (f_1 - \delta_1(s, s''), f_2 - \delta_2(s, s''), f_3 + \delta_1(s, s'') + \delta_2(s, s''))$ and a directed (s, s'')-path in $G_{\mathcal{G}_\epsilon}$ whose first arc is labelled with p_i, it holds $\omega_i(s) < \omega_i(s'')$ if and only if*

$$f_2 < \delta_1(s, s'') + \delta_2(s, s''), \text{ when } p_i \text{ migrates in } s \text{ from } \Pi_2 \text{ to } \Pi_3; \quad (1)$$

$$f_2 < f_1 + \delta_2(s, s''), \text{ when } f_2 > f_1 + \epsilon \text{ and } p_i \text{ migrates in } s \text{ from } \Pi_2 \text{ to } \Pi_1; \quad (2)$$

$$f_1 < \delta_1(s, s'') + \delta_2(s, s''), \text{ when } p_i \text{ migrates in } s \text{ from } \Pi_1 \text{ to } \Pi_3; \quad (3)$$

$$f_1 < f_2 + \delta_1(s, s''), \text{ when } f_1 > f_2 + \epsilon \text{ and } p_i \text{ migrates in } s \text{ from } \Pi_1 \text{ to } \Pi_2. \quad (4)$$

Proof. The claim follows directly by evaluating the two payoffs $\omega_i(s)$ and $\omega_i(s'')$ in all the four cases. □

We can now characterize the set of Second Order Nash equilibria for $\mathcal{G}_\epsilon$.

Theorem 2. *$N(\mathcal{G}_\epsilon) = N^0(\mathcal{G}_\epsilon) \cup \{((\ell + 3j)\epsilon, \ell\epsilon, 1 - (2\ell + 3j)\epsilon), (\ell\epsilon, (\ell + 3j)\epsilon, 1 - (2\ell + 3j)\epsilon) | j = 0, \ldots, \lfloor \frac{1-4\epsilon}{3\epsilon} \rfloor$ and $\ell = 3, \ldots, \lfloor \frac{1-3j\epsilon}{2\epsilon} \rfloor\}$.*

Proof. Since no improving step exists in which a player leaves Π_3, we can use Lemma 4 to show whether or not a state s is a Second Order Nash equilibrium. In particular, we will show that s is a Second Order Nash equilibrium by comparing it with all the Second Order Nash equilibria $s'' \in \rho(s')^{k^*}$ for any player $p_i \in P(s)$ and $s' \in \rho(s)_i^1$, while we will show that s cannot be a Second Order Nash equilibrium by providing a player $p_i \in P(s)$, a state $s' \in \rho(s)_i^1$ and a Second Order Nash equilibria $s'' \in \rho(s')^{k^*}$ for which the related inequality given in Lemma 4 is violated. Consider a state $s = (f_1, f_2, f_3) \in S$. We will first analyze all the states having $f_1 \leq 2\epsilon$ or $f_2 \leq 2\epsilon$.

- Case $f_1 \geq \epsilon$ and $f_2 = 0$. Choose a player p_i routing on Π_1 and let s'' be any Second Order Nash equilibrium such that $s'' \in \rho(s')^{k^*}$ where $s' \in \rho(s)_i^1$ is the state obtained from s when p_i migrates from Π_1 to Π_3. Since $\delta_2(s, s'') \leq 0$, the condition $f_1 < \delta_1(s, s'')$ has to be satisfied when inequality (3) holds. But, since $f_1 - \delta_1(s, s'') \geq 0$ by definition, such a condition can never hold. Hence the only Second Order Nash equilibria having $f_2 = 0$ are the two Nash equilibria $(0, 0, 1)$ and $(\epsilon, 0, 1 - \epsilon)$.
- Case $f_1 = 0$ and $f_2 \geq \epsilon$. By a symmetric argument, the only Second Order Nash equilibria having $f_1 = 0$ are the two Nash equilibria $(0, 0, 1)$ and $(0, \epsilon, 1 - \epsilon)$.
- Case $f_1 \geq \epsilon$ and $f_2 = \epsilon$. Choose a player p_i routing on Π_1 and let s'' be any Second Order Nash equilibrium such that $s'' \in \rho(s')^{k^*}$ where $s' \in \rho(s)_i^1$ is the state obtained from s when p_i migrates from Π_1 to Π_3. For any such an s'' it must hold $\delta_1(s, s'') < f_1$ and $\delta_2(s, s'') \leq 0$, since it is not possible to reach any of the states $(\epsilon, 0, 1 - \epsilon)$ and $(0, \epsilon, 1 - \epsilon)$ without passing through the state $(\epsilon, \epsilon, 1 - 2\epsilon)$. We have $\delta_1(s, s'') + \delta_2(s, s'') < f_1$ thus inequality (3) can never be satisfied. Hence the only Second Order Nash equilibrium having $f_2 = \epsilon$ is the Nash equilibrium $(\epsilon, \epsilon, 1 - 2\epsilon)$.
- Case $f_1 = \epsilon$ and $f_2 \geq \epsilon$. By a symmetric argument, the only Second Order Nash equilibrium having $f_1 = \epsilon$ is the Nash equilibrium $(\epsilon, \epsilon, 1 - 2\epsilon)$.

- Case $f_1 \geq 2\epsilon$ and $f_2 = 2\epsilon$. Choose a player p_i routing on Π_1 and let s'' be any Second Order Nash equilibrium such that $s'' \in \rho(s')^{k^*}$ where $s' \in \rho(s)_i^1$ is the state obtained from s when p_i migrates from Π_1 to Π_3. Again, for any such an s'' it must hold $\delta_1(s, s'') \leq f_1 - \epsilon$ and $\delta_2(s, s'') \leq \epsilon$, since it is not possible to reach any of the states $(\epsilon, 0, 1-\epsilon)$ and $(0, \epsilon, 1-\epsilon)$ without passing through the state $(\epsilon, \epsilon, 1-2\epsilon)$. We have $\delta_1(s, s'') + \delta_2(s, s'') \leq f_1$ thus inequality (3) can never be satisfied. Hence no Second Order Nash equilibria may exist having $f_2 = 2\epsilon$.
- Case $f_1 = 2\epsilon$ and $f_2 \geq 2\epsilon$. By a symmetric argument, no Second Order Nash equilibria may exist having $f_1 = 2\epsilon$.

We show the general claim for the remaining states by induction on j. The basic step is when $j = 0$. By induction on ℓ we prove that all the states $s_\ell = (\ell\epsilon, \ell\epsilon, 1 - 2\ell\epsilon) \in N(\mathcal{G}_\epsilon)$. Consider first of all the state $s_3 = (3\epsilon, 3\epsilon, 1 - 6\epsilon)$. Starting from s_3, for any $p_i \in P(s_3)$ migrating to Π_3 from either Π_1 or Π_2, it is possible to create a path of improving steps reaching the state $s'' = (\epsilon, \epsilon, 1 - 2\epsilon)$ and not passing through any other Second Order Nash equilibrium (since all the states in the path would have either $f_1 \leq 2\epsilon$ or $f_2 \leq 2\epsilon$). Since in any case $\delta_1(s_3, s'') + \delta_2(s_3, s'') = 4\epsilon > 3\epsilon$, inequalities (1) and (3) are both satisfied, thus $s_3 \in N(\mathcal{G}_\epsilon)$.

Now suppose, for the sake of induction, that $s_r = (r\epsilon, r\epsilon, 1 - 2r\epsilon) \in N(\mathcal{G}_\epsilon)$ for any $3 \leq r < \ell$ and consider the state s_ℓ. Let us assume for a while that starting from s_ℓ, for any $p_i \in P(s_\ell)$ migrating to Π_3 from either Π_1 or Π_2, it is possible to create a path of improving steps reaching the state $s'' = (\epsilon, \epsilon, 1 - 2\epsilon)$ and not passing through any other Second Order Nash equilibrium. Since in any case $\delta_1(s_\ell, s'') + \delta_2(s_\ell, s'') = 2(\ell - 1)\epsilon > \ell\epsilon$, we have $s_\ell \in N(\mathcal{G}_\epsilon)$ provided such a desired path can be obtained. This task is achieved by considering a sequence of migrations towards Π_3 so that at each state of the path the absolute value of the difference between the flow routed on Π_1 and the one routed on Π_2 is equal to either ϵ or 2ϵ. Any such state cannot be a Second Order Nash equilibrium. In fact, a state $s = (f_s + \epsilon, f_s, 1 - 2f_s - \epsilon)$ with $f_s + \epsilon \leq \ell\epsilon$ cannot belong to $N(\mathcal{G}_\epsilon)$ since by the inductive hypothesis, the state $s' = (f_s, f_s, 1 - 2f_s) \in N(\mathcal{G}_\epsilon)$, $s' \in \rho(s)_i^1$ for some $p_i \in P(s)$ and $\omega_i(s) > \omega_i(s')$. Analogously, a state $s = (f_s + 2\epsilon, f_s, 1 - 2f_s - 2\epsilon)$ with $f_s + 2\epsilon \leq \ell\epsilon$ cannot belong to $N(\mathcal{G}_\epsilon)$ since by the inductive hypothesis, the state $s' = (f_s + \epsilon, f_s + \epsilon, 1 - 2f_s - 2\epsilon) \in N(\mathcal{G}_\epsilon)$, $s' \in \rho(s)_i^1$ for some $p_i \in P(s)$ and $\omega_i(s) > \omega_i(s')$. By symmetry, also the states $(f_s, f_s + \epsilon, 1 - 2f_s - \epsilon)$ and $(f_s, f_s + 2\epsilon, 1 - 2f_s - 2\epsilon)$ cannot belong to $N(\mathcal{G}_\epsilon)$. This completes the proof of the basic step.

Now suppose for the sake of induction that the claim holds for any $0 \leq p < j$, we have to show that it holds also for the value j and for any $\ell = 3, \ldots, \frac{1-3j\epsilon}{2\epsilon}$. We prove it by induction on ℓ.

We have proved that for any state $s = (f_1, f_2, f_3)$, if $0 < |f_1 - f_2| < 3\epsilon$ then $s \notin N(\mathcal{G}_\epsilon)$. Suppose that for any $0 \leq p < j$, the state $((\ell + 3p)\epsilon, \ell\epsilon, 1 - (2\ell + 3p)\epsilon)$ belongs to $N(\mathcal{G}_\epsilon)$. It is not difficult to see that using the same argument exploited above together with the inductive hypothesis, it is possible to prove that for any state $s = (f_1, f_2, f_3)$, if $3p\epsilon < |f_1 - f_2| < 3(p+1)\epsilon$, then $s \notin N(\mathcal{G}_\epsilon)$.

The basic step is when $\ell = 3$. Consider the state $s = (f_1, f_2, f_3) = ((3 + 3j)\epsilon, 3\epsilon, 1 - (6 + 3j)\epsilon)$. We have to distinguish among three possible states $s' \in \rho(s)^1$.

1. s' is the state obtained from s when a player p_i migrates from Π_1 to Π_2. Such a state $s' = (f_1', f_2', f_3')$ is such that $f_1' - f_2' = (2 + 3j)\epsilon - 4\epsilon = (3j - 2)\epsilon = (3(j-1) + 1)\epsilon$. Thus, since $3(j-1)\epsilon < f_1' - f_2' < 3j\epsilon$, we have $s' \notin N(\mathcal{G}_\epsilon)$. Now consider the sequence of improving steps starting from s' obtained by applying the following rule: *"While the flow routed on Π_2 is strictly greater than ϵ, let a player migrate from Π_2 to Π_3 and then let a player migrate from Π_1 to Π_3"*. By doing so, the difference between the flow routed on Π_1 and that routed on Π_2 in any state of the sequence will always be equal to either $(3(j-1)+1)\epsilon$ or $(3(j-1)+2)\epsilon$, thus the created path can never reach a Second Order Nash equilibrium. When the flow routed on Π_2 reaches the value ϵ we can consider a sequence of migrations from Π_1 to Π_3 until the state $s'' = (\epsilon, \epsilon, 1 - 2\epsilon)$ is reached. Again, no other Second Order Nash equilibrium is traversed during these migrations. In this case, since $f_1 = (3 + 3j)\epsilon$, $f_2 = 3\epsilon$ and $\delta_1(s, s'') = (2 + 3j)\epsilon$, inequality (4) is satisfied.
2. s' is the state obtained from s when a player p_i migrates from Π_1 to Π_3. Such a state $s' = (f_1', f_2', f_3')$ is such that $f_1' - f_2' = (2 + 3j)\epsilon - 3\epsilon = (3j - 1)\epsilon = (3(j-1) + 2)\epsilon$. Thus, since $3(j-1)\epsilon < f_1' - f_2' < 3j\epsilon$, we have $s' \notin N(\mathcal{G}_\epsilon)$. Now consider the sequence of improving steps starting from s' obtained by applying the following rule: *"While the flow routed on Π_2 is strictly greater than ϵ, let a player migrate from Π_1 to Π_3 and then let a player migrate from Π_2 to Π_3"*. By doing so, the difference between the flow routed on Π_1 and that routed on Π_2 in any state of the sequence will always be equal to either $(3(j-1)+1)\epsilon$ or $(3(j-1)+2)\epsilon$, thus the created path can never reach a Second Order Nash equilibrium. When the flow routed on Π_2 reaches the value ϵ we can consider a sequence of migrations from Π_1 to Π_3 until the state $s'' = (\epsilon, \epsilon, 1 - 2\epsilon)$ is reached. Again, no other Second Order Nash equilibrium is traversed during these migrations. In this case, since $f_1 = (3 + 3j)\epsilon$, $\delta_1(s, s'') = (2 + 3j)\epsilon$, and $\delta_2(s, s'') = 2\epsilon$ inequality (3) is satisfied.
3. s' is the state obtained from s when a player p_i migrates from Π_2 to Π_3. Such a state $s' = (f_1', f_2', f_3')$ is such that $f_2' = 2\epsilon$ then $s' \notin N(\mathcal{G}_\epsilon)$ since we have proved that no Second Order Nash equilibria exist whose flow routed on Π_2 is equal to 2ϵ. Again, we can let one of the two players routing on Π_2 migrate to Π_3 and then consider a sequence of migrations from Π_1 to Π_3 until the state $s'' = (\epsilon, \epsilon, 1 - 2\epsilon)$ is reached. In this case, since $f_2 = 3\epsilon$, $\delta_1(s, s'') = (2 + 3j)\epsilon$ and $\delta_2(s, s'') = 2\epsilon$, inequality (1) is satisfied.

Thus the basic step is proved. Now, we show the inductive step. Assume that the claim holds for any r such that $3 \leq r < \ell$ and consider the state $s = (f_1, f_2, f_3) = ((\ell + 3j)\epsilon, \ell\epsilon, 1 - (2\ell + 3j)\epsilon)$. We have to distinguish among three possible states $s' \in \rho(s)^1$.

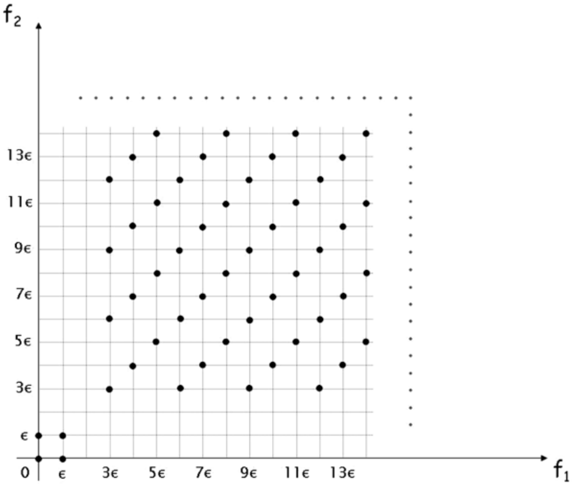


Fig. 4. Second Order Nash equilibria for $\mathcal{G}_\epsilon$.

1. s' is the state obtained from s when a player p_i migrates from Π_1 to Π_2. Such a state $s' = (f_1', f_2', f_3')$ is such that $f_1' - f_2' = (\ell + 3j - 1)\epsilon - (\ell + 1)\epsilon = (3j - 2)\epsilon = (3(j - 1) + 1)\epsilon$. Thus, since $3(j - 1)\epsilon < f_1' - f_2' < 3j\epsilon$, we have $s' \notin N(\mathcal{G}_\epsilon)$. Now consider the sequence of improving steps starting from s' obtained by applying the following rule: *"While the flow routed on Π_2 is strictly greater than ϵ, let a player migrate from Π_2 to Π_3 and then let a player migrate from Π_1 to Π_3"*. By doing so, the difference between the flow routed on Π_1 and that routed on Π_2 in any state of the sequence will always be equal to either $(3(j - 1) + 1)\epsilon$ or $(3(j - 1) + 2)\epsilon$, thus the created path can never reach a Second Order Nash equilibrium. When the flow routed on Π_2 reaches the value ϵ we can consider a sequence of migrations from Π_1 to Π_3 until the state $s'' = (\epsilon, \epsilon, 1 - 2\epsilon)$ is reached. Again, no other Second Order Nash equilibrium is traversed during these migrations. In this case, since $f_1 = (\ell + 3j)\epsilon, f_2 = \ell\epsilon$ and $\delta_1(s, s'') = (\ell + 3j - 1)\epsilon$, inequality (4) is satisfied.
2. s' is the state obtained from s when a player p_i migrates from Π_1 to Π_3. Such a state $s' = (f_1', f_2', f_3')$ is such that $f_1' - f_2' = (\ell + 3j - 1)\epsilon - \ell\epsilon = (3j - 1)\epsilon = (3(j - 1) + 2)\epsilon$. Thus, since $3(j - 1)\epsilon < f_1' - f_2' < 3j\epsilon$, we have $s' \notin N(\mathcal{G}_\epsilon)$. Now consider the sequence of improving steps starting from s' obtained by applying the following rule: *"While the flow routed on Π_2 is strictly greater than ϵ, let a player migrate from Π_1 to Π_3 and then let a player migrate from Π_2 to Π_3"*. By doing so, the difference between the flow routed on Π_1 and that routed on Π_2 in any state of the sequence will always be equal to either $(3(j - 1) + 1)\epsilon$ or $(3(j - 1) + 2)\epsilon$, thus the created path can never reach a Second Order Nash equilibrium. When the flow routed on Π_2 reaches the value ϵ we can consider a sequence of migrations from Π_1 to Π_3 until the state $s'' = (\epsilon, \epsilon, 1 - 2\epsilon)$ is reached. Again, no other Second Order Nash equilibrium is traversed during these migrations. In this case, since $f_1 = (\ell + 3j)\epsilon$, $\delta_1(s, s'') = (\ell + 3j - 1)\epsilon$ and $\delta_2(s, s'') = (\ell - 1)\epsilon$ inequality (3) is satisfied.
3. s' is the state obtained from s when a player p_i migrates from Π_2 to Π_3. Such a state $s' = (f_1', f_2', f_3')$ is such that $f_1' - f_2' = (\ell + 3j)\epsilon - (\ell - 1)\epsilon = (3j + 1)\epsilon$. We have $s' \notin N(\mathcal{G}_\epsilon)$. In fact, the state $\bar{s} = ((\ell + 3j - 1)\epsilon, (\ell - 1)\epsilon, f_3')$ obtained from s' by letting a player p_i migrate from Π_1 to Π_3 is a Second Order Nash equilibrium by the inductive hypothesis, since $\bar{s} = ((p + 3j)\epsilon, p\epsilon, f_3'')$ with $p = \ell - 1$. Now consider the state $\tilde{s} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ obtained from s' by letting a player migrate from Π_1 to Π_2. We have $\tilde{f}_1 - \tilde{f}_2 = (\ell + 3j - 1)\epsilon - \ell\epsilon = (3j - 1)\epsilon = (3(j - 1) + 2)\epsilon$ thus falling into case 2. This means that there exists a path of improving steps ending to the state $(\epsilon, \epsilon, 1 - 2\epsilon)$. In this case, since $f_2 = \ell\epsilon$, $\delta_1(s, s'') = (\ell + 3j - 1)\epsilon$ and $\delta_2(s, s'') = (\ell - 1)\epsilon$ inequality (1) is satisfied.

We have shown that $s \in N(\mathcal{G}_\epsilon)$ and the inductive step is proved. By symmetry, the same result holds also for $s = (\ell\epsilon, (\ell + 3j)\epsilon, 1 - (2\ell + 3j)\epsilon)$. This completes the proof since we have also shown that no other state can belong to $N(\mathcal{G})$. □

For the ease of understanding, the structure of $N(\mathcal{G}_\epsilon)$ has been depicted in Fig. 4.
As a consequence of Theorem 2, the following corollary holds.

Corollary 2. *The price of stability of the Second Order Nash equilibrium for $\mathcal{G}_\epsilon$ is* 1.

In this problem, as well as in other non-trivial multiplayer games, we have seen that the structure of the state graph can be really intricate thus making the definition of Second Order Nash equilibria a very challenging task. In order to ease the computation one can work on a simplified version of the state graph by considering, for example, the existence of a particular ordering in which improving steps are performed during the evolution of the game. Such a scenario seems perfectly reasonable and different ordering strategies can be considered indeed as some sort of coordination mechanisms [14] that can be adopted during the game in order to lead the players towards a desired behavior.

To this aim consider the following ordering mechanism $\mathcal{M}$: *at each state s, the player $p_i \in P(s)$ using the most congested path is the one allowed to change her strategy (breaking ties arbitrarily).*

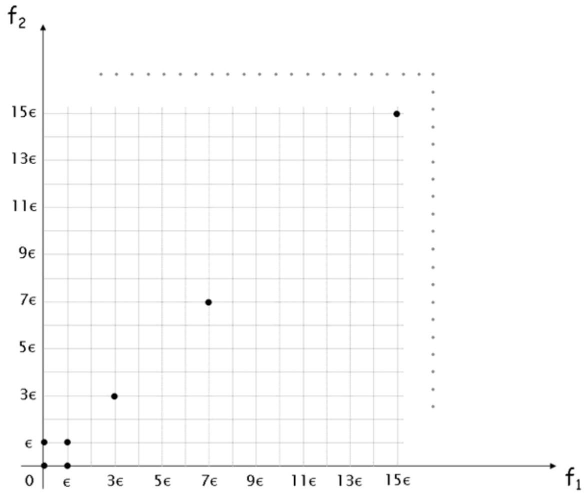


Fig. 5. Second Order Nash equilibria for $\mathcal{G}_\epsilon$ when mechanism $\mathcal{M}$ is adopted.

In order to show Theorem 3, we first prove the following simple lemma.

Lemma 5. *Let Π be a path on $G_{\mathcal{G}_\epsilon}$ in which each improving step consists in a migration towards Π_3 and let $\omega_{\Pi_r}(s)$ be the payoff of any player routing his traffic on Π_r in any state $s \in S$ with $r = 1, 2, 3$. Then for any two states $s = (f_1, f_2, f_3)$ and $s' = (f_1 - \delta_1(s, s'), f_2 - \delta_2(s, s'), f_3 + \delta_1(s, s') + \delta_2(s, s')) \in \Pi$, it holds $\omega_{\Pi_3}(s') - \omega_{\Pi_r}(s) = \delta_1(s, s') + \delta_2(s, s') - f_r$, for $r = 1, 2$.*

Proof. We have $\omega_{\Pi_r}(s) = 1 + f_r + f_3$ while $\omega_{\Pi_3}(s') = f_1 + f_2 + 2f_3 + \delta_1(s, s') + \delta_2(s, s')$. $\square$

We can state the following result.

Theorem 3. *When mechanism $\mathcal{M}$ is adopted $N(\mathcal{G}_\epsilon) = N^0(\mathcal{G}_\epsilon) \cup \{((2^{i+1} - 1)\epsilon, (2^{i+1} - 1)\epsilon, 1 - 2(2^{i+1} - 1)\epsilon) | i = 1, \ldots \lfloor \log_2(\frac{1+2\epsilon}{2\epsilon}) \rfloor - 1\}$.*

Proof. Let $F = N^0(\mathcal{G}_\epsilon) \cup \{((2^{i+1} - 1)\epsilon, (2^{i+1} - 1)\epsilon, 1 - 2(2^{i+1} - 1)\epsilon) | i = 1, \ldots \lfloor \log_2(\frac{1+2\epsilon}{2\epsilon}) \rfloor - 1\}$. Consider a state $s = (f_s, f_s, 1 - 2f_s) \in F$ with $f_s < 1/4$ and the state $s' = (2f_s + \epsilon, 2f_s + \epsilon, 1 - 4f_s - 2\epsilon) \in F$. We claim that if $s \in N(\mathcal{G}_\epsilon)$ then also $s' \in N(\mathcal{G}_\epsilon)$. By applying mechanism $\mathcal{M}$, we have that $s \in \rho(s')^{k^*}$ and the sequence of related improving steps satisfies the conditions of Lemma 5. Without loss of generality we can suppose that the player p_i performing the improving step from s' leaves Π_1 in favor of Π_3. We have $\omega_i(s') = 2 - 2f_s - \epsilon < \omega_i(s) = 2 - 2f_s$. Applying Lemma 5, we have that no other state s'' may exist on the path between s' and s in $G_{\mathcal{G}_\epsilon}$ such that $\omega_i(s'') < \omega_i(s)$. Hence $s' \in N(\mathcal{G}_\epsilon)$.

We now prove by induction that $F \subseteq N(\mathcal{G}_\epsilon)$. Assume that for the value i the state $((2^{i+1} - 1)\epsilon, (2^{i+1} - 1)\epsilon, 1 - 2(2^{i+1} - 1)\epsilon) \in N(\mathcal{G}_\epsilon)$. By using the result just shown above, we have that $(2(2^{i+1} - 1)\epsilon + \epsilon, 2(2^{i+1} - 1)\epsilon + \epsilon, 1 - 4(2^{i+1} - 1)\epsilon - 2\epsilon) = ((2^{i+2} - 1)\epsilon, (2^{i+2} - 1)\epsilon, 1 - 2(2^{i+2} - 1)\epsilon) \in N(\mathcal{G}_\epsilon)$ thus showing the claim for the value $i + 1$. Since for $i = 0$ the state $(\epsilon, \epsilon, 1 - 2\epsilon) \in N^0(\mathcal{G}_\epsilon) \subseteq F$, the inductive argument holds.

Now, in order to show that $F = N(\mathcal{G}_\epsilon)$, consider a state $s = (f_1, f_2, f_3) \notin F$ such that no other state $s' \in N(\mathcal{G}_\epsilon) \setminus F$ exists on any path of $G_{\mathcal{G}_\epsilon}$ starting from s. Suppose without loss of generality $f_1 > f_2$. By applying mechanism $\mathcal{M}$, we have that the state $s' = (f_2, f_2, 1 - 2f_2) \in \rho(s)^{k^*}$. If $s' \in F$ we have $\omega_i(s) = 1 + f_1 + f_3 = 2 - f_2 \geq 2 - 2f_2 = \omega_i(s')$, hence s cannot be a Second Order Nash equilibrium. If $s \notin F$, a state $\bar{s} = (f_{\bar{s}}, f_{\bar{s}}, 1 - 2f_{\bar{s}}) \in F$ is reached such that $f_{\bar{s}} \geq \frac{f_2}{2}$. By applying again Lemma 5, we have that the maximum value of $\omega_i(\bar{s})$ is achieved when $f_{\bar{s}} = \frac{f_2}{2}$. We thus have $\omega_i(s) = 1 + f_1 + f_3 = 2 - f_2 \geq \omega_i(\bar{s})$, hence again s cannot be a Second Order Nash equilibrium. $\square$

In Fig. 5 we show the structure of $N(\mathcal{G}_\epsilon)$ in this case. It can be easily noted that the set of Second Order Nash equilibria is a subset of the one achieved in the general case.

Corollary 3. *When mechanism $\mathcal{M}$ is adopted the price of stability of the Second Order Nash equilibrium of $\mathcal{G}_\epsilon$ falls in the interval $[1; \frac{13}{12}]$.*

Proof. If $\log_2(\frac{1+2\epsilon}{2\epsilon})$ is an integral number we have that $(2^{\lfloor \log_2(\frac{1+2\epsilon}{2\epsilon}) \rfloor} - 1)\epsilon = \frac{1}{2}$, hence the optimal state $(\frac{1}{2}, \frac{1}{2}, 0) \in N(\mathcal{G}_\epsilon)$ thus proving the lower bound on the price of stability of the Second Order Nash equilibrium. On the other hand, the worst case occurs when the state $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) \in N(\mathcal{G}_\epsilon)$. In this case we have that the cost of the solution is $\frac{13}{8}$ which compared with the optimal cost of $\frac{3}{2}$ gives the upper bound of $\frac{13}{12}$ on the price of stability. $\square$

Thus, on the one hand the introduction of the ordering mechanism $\mathcal{M}$ has simplified the set of Second Order Nash equilibria, on the other hand it has (slightly) worsened its price of stability.

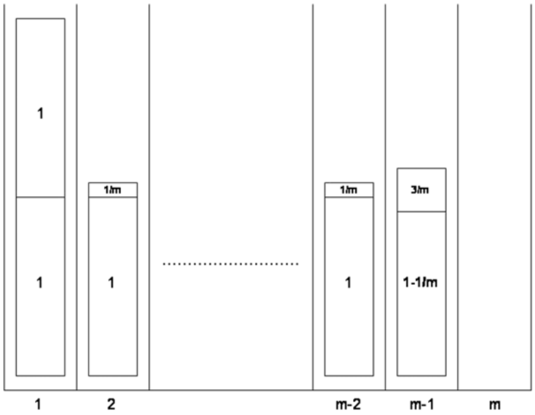


Fig. 6. An instance of the load balancing game.

Letting ϵ go to zero we obtain the general version of Braess's Paradox in the *Wardrop* model. We argue here that the results of Corollaries 2 and 3 still hold in this case, since, when ϵ tends to zero, $\ell\epsilon = \frac{1}{2}$ when $j = 0$ and $\ell = \lfloor \frac{1-3j\epsilon}{2\epsilon} \rfloor$ in the claim of Theorem 2 and $(2^{\lfloor \log_2(\frac{1+2\epsilon}{2\epsilon}) \rfloor} - 1)\epsilon \in [\frac{1}{4}; \frac{1}{2}]$ in the claim of Theorem 3 respectively.

4.2. Selfish load balancing

In this section we analyze the applications of Second Order Nash equilibria to the load balancing game, the special case of the *KP* model in which all the machines have identical speed.

Given a state s, let $L_j(s)$ be the load of machine j in s, we define $T(s) = \max_{j=1,\dots,m} L_j(s)$ and $t(s) = \min_{j=1,\dots,m} L_j(s)$ as the maximum and the minimum completion time respectively of the m machines in state s. For any (s, s')-path π in $G_{\mathcal{G}}$, let $M(\pi)$ be the set of machines involved in the improving steps defining such a path. We denote as $T_{M(\pi)}(s)$ (resp. $t_{M(\pi)}(s)$) the maximum (resp. minimum) completion time on the machines belonging to $M(\pi)$ in state s.

The following lemma is just a restatement of two well-known properties of improving steps in the *KP* model first shown in [20,18], respectively.

Lemma 6. *For any (s, s')-path π in $G_{\mathcal{G}}$ it holds $T_{M(\pi)}(s) \geq T_{M(\pi)}(s')$ and $t_{M(\pi)}(s) \leq t_{M(\pi)}(s')$.*

We show that no proper Second Order Nash equilibrium exists for the load balancing game, i.e., $N(\mathcal{G})$ collapses to the set of Nash equilibria.

Theorem 4. *Let $\mathcal{G}$ be any load balancing game, it holds $N(\mathcal{G}) = N^0(\mathcal{G})$.*

Proof. For the sake of contradiction, consider a state $s \in N(\mathcal{G}) \setminus N^0(\mathcal{G})$. Clearly, we have $P(s) \neq \emptyset$. Consider a player $p_i \in P(s)$ using a machine j in s such that $L_j(s)$ is maximal, let $s' \in \rho(s)_i^1$ be the state reached when p_i performs the improving step of leaving machine j in favor of a machine j' such that $L_{j'}(s) = t(s)$ and consider a state $s'' \in \rho(s')^{k^*}$. By applying Lemma 6, we have $t(s) = t_{M(s,s'')}(s) \leq t_{M(s,s'')}(s'')$, thus no machine $j'' \in M(s, s'')$ exists such that $L_{j''}(s) > L_j(s)$ because otherwise there would have been an improving step for some players using machine j'' in s thus contradicting the maximality of $L_j(s)$. This shows that $L_j(s) = T_{M(s,s'')}(s) \geq T_{M(s,s'')}(s'')$. Since we have $\omega_i(s) = T_{M(s,s'')}(s) \geq T_{M(s,s'')}(s'') \geq \omega_i(s'')$, s cannot be a proper Second Order Nash equilibrium. □

As a consequence of the above theorem, for this model we are in a situation in which there is no need to study the performances of Second Order Nash equilibria since they are just Nash equilibria. However, as we have seen in Proposition 2, convergence towards these equilibria may be affected. Moreover, as we illustrate in the following example, since Second Order improving steps are based on a comparison of the current state with the worst one which can be reached at the end of the evolution, the use extended rationality can potentially lead the game towards better Nash equilibria.

Example 1. Consider an instance of the load balancing game in which we have m machines and $2m - 2$ jobs. In particular, there are $m - 1$ jobs of length 1, one job of length $1 - \frac{1}{m}$, one job of length $\frac{3}{m}$ and $m - 3$ jobs of length $\frac{1}{m}$. Now, suppose that the game is in the state represented in Fig. 6 and that the following sequence of improving steps takes place: the job of length $1 - \frac{1}{m}$ migrates to machine m, all the $m - 3$ jobs of length $\frac{1}{m}$ migrate to machine $m - 1$ thus filling it up to the value 1 and finally a job of length 1 migrates from machine 1 to machine m thus leading the game to the state depicted in Fig. 7. Such a state is the worst Nash equilibrium and yields a makespan equal to $2 - \frac{1}{m}$. The best Nash equilibrium (and optimal solution for the underlying optimization problem), depicted in Fig. 8, yields a makespan equal to $1 + \frac{2}{m}$.

We show that the use of extended rationality can avoid such an undesired convergence to the worst Nash equilibrium. Let s be the state depicted in Fig. 6, s' be the worst Nash equilibrium and p_i be the player owning the job of length $1 - \frac{1}{m}$. By using the sequence of migrations described above, we have that $s' \in \rho(s)_i^{k^*}$ and $\omega_i(s) < \omega_i(s')$. Thus, even though s is not a Second Order Nash equilibrium, we have that, by applying her extended rationality, p_i will renounce to perform her

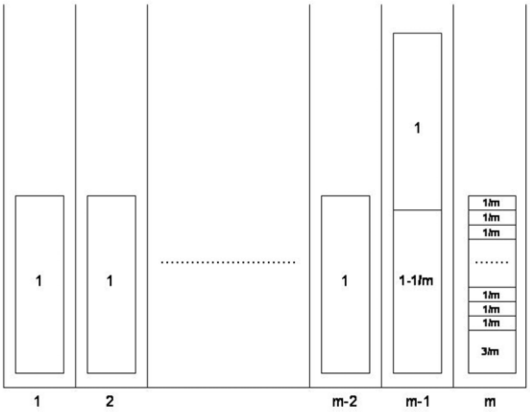


Fig. 7. The worst Nash equilibrium.

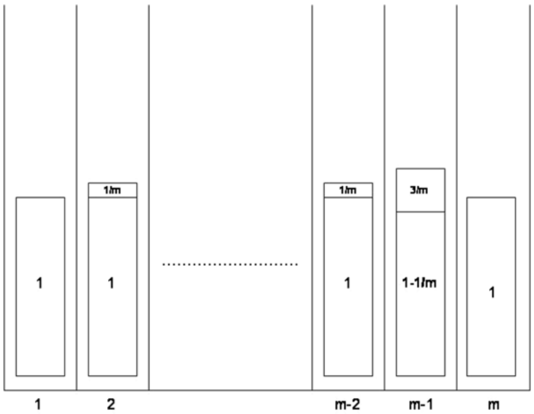


Fig. 8. The best Nash equilibrium.

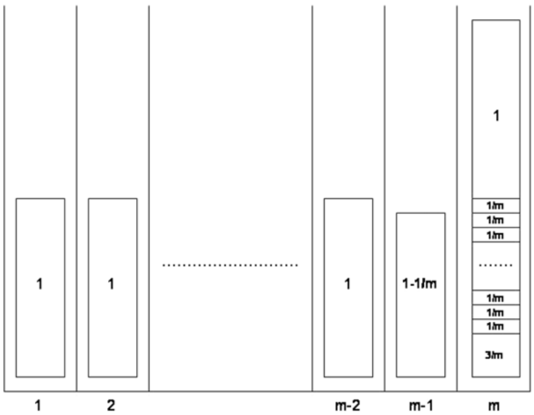


Fig. 9. A "bad" instance of the load balancing game.

improving step when the game is on state *s*. A similar reasoning can be applied also to the players owning the jobs of length $\frac{3}{m}$ and $\frac{1}{m}$, thus showing that several improving steps leaving state *s* does not belong to the Second Order state graph of the game. It is not difficult to see that, if players act this way, the game will finally reach the best Nash equilibrium. There is, indeed, still one case (depicted in Fig. 9) in which neither the use of the extended rationality can prevent the game from reaching the worst Nash equilibrium. This happens when the player owning the job of length 1 migrates from machine m to machine $m - 1$.

However, this bad situation could be avoided if such a player renounced to her improving step conscious of the fact that this would lead the game towards a better outcome for her. This assumption introduces a further extension in our definition of rational agents. We formalize this idea in the next section.

5. Other notions of selfish farsighted behavior

What we have seen so far are agents always comparing the current situation with the worst Second Order equilibrium the game can reach after their improving steps. We will call these agents *prudent agents* because if there is a chance of worsening their payoffs they will stay quiet and will not perform any improving step. Clearly, it is also possible to consider *rash agents* choosing to perform improving steps when there is a chance of reaching a good equilibrium for them, thus comparing with the best Second Order equilibrium. When considering rash agents, the definition of Second Order Nash equilibria becomes the following.

Definition 3 (*Rash Agents*). Let $\mathcal{G}$ be a convergent game. The set $N^k(\mathcal{G}) = \{s \in S : \forall p_i \in P(s)$ and $\forall s' \in \rho(s)^1_i$ and $\forall s'' \in N^k(\mathcal{G})$ such that $s'' \in \rho_{N^k(\mathcal{G})}(s')^k$ it holds $\omega_i(s) \leq \omega_i(s'')\}$ is the set of all the Second Order k-Nash equilibria of game $\mathcal{G}$, for any integer $k \geq 0$.

As we have seen at the end of the previous section, extended rationality can also mean that a player eventually renounce to perform an improving step if she can benefit from such a choice. Thus, if Definitions 1 and 3 model equilibria for prudent and rash agents respectively, we now propose other two definitions that will characterize equilibria for two classes of even more farsighted agents that we will call patient prudent and patient rash agents respectively. To this aim, we first need to introduce the following notation. For any set of equilibrium states $E \subseteq S$, let $\rho_{s',E}(s)^k$ be the set of states which can be reached from s in k steps in $G_{\mathcal{G}}$ without using the edge (s, s'), $worst_i(E, s') = \max_{s'' \in E : s'' \in \rho_E(s')^k}\{\omega_i(s'')\}$ be the worst payoff for player p_i yielded by any equilibrium in E which can be reached in k steps starting from s' and $worst_i(E, s, s') = \max_{s'' \in E : s'' \in \rho_{s',E}(s)^k}\{\omega_i(s'')\}$ be the worst payoff for player p_i yielded by any equilibrium in E which can be reached in k steps starting from s without using the edge (s, s'). Analogously, define $best_i(E, s') = \min_{s'' \in E : s'' \in \rho_E(s')^k}\{\omega_i(s'')\}$ and $best_i(E, s, s') = \min_{s'' \in E : s'' \in \rho_{s',E}(s)^k}\{\omega_i(s'')\}$. Considering patient prudent agents and patient rash agents respectively, the definitions of the set of Second Order Nash equilibria become the following ones.

Definition 4 (*Patient Prudent Agents*). Let $\mathcal{G}$ be a convergent game. The set $N^k(\mathcal{G}) = \{s \in S : \forall p_i \in P(s)$ such that $worst_i(N^k(\mathcal{G}), s') \leq worst_i(N^k(\mathcal{G}), s, s')\}$ is the set of all the Second Order k-Nash equilibria of game $\mathcal{G}$, for any integer $k \geq 0$.

Definition 5 (*Patient Rash Agents*). Let $\mathcal{G}$ be a convergent game. The set $N^k(\mathcal{G}) = \{s \in S : \forall p_i \in P(s)$ such that $best_i(N^k(\mathcal{G}), s') < best_i(N^k(\mathcal{G}), s, s')\}$ is the set of all the Second Order k-Nash equilibria of game $\mathcal{G}$, for any integer $k \geq 0$.

6. Conclusions

In this paper we tried to give an impulse towards the definition of a better model for selfish rational agents by exploiting a simple and intuitive observation. If starting from a state s in which a set of players $P(s)$ are unhappy of their payoffs, the game can reach an equilibrium s' in which for any $p_i \in P(s)$ it holds $\omega_i(s) < \omega_i(s')$, then it is rational to consider s as an equilibrium state, since all players who have an incentive in deviating from s discover that such an incentive is just illusory. This view point leads us to Second Order Nash equilibria which, in spite of a simple intuitive nature, required not trivial arguments in order to be captured in a formal definition.

A well-studied problem in Game Theory has been that of reducing the set of Nash equilibria of a game by eliminating those which can be considered in some sense "irrational". This process can surely provide an improvement on the price of anarchy of Nash equilibria. However, no benefits can be achieved in all those games in which the price of stability of Nash equilibria is too high. Our definition of Second Order Nash equilibria has the property of expanding the set of Nash equilibria thus being able to potentially improve on the price of stability.

We provided different types of applications of Second Order Nash equilibria to games such as the Prisoner's Dilemma, the *Wardrop* model and the *KP* model. We believe that our work can open a new window on this young and fascinating research field, by widening the notion of rationality of players.

A lot of open questions are thus introduced by this vision. The first one is certainly that of giving further validation of Second Order equilibria by using them together with other known equilibria and presenting good applications. To this aim, the definition of Second Order Sink equilibria seems to be a promising research direction. Moreover, there is the important issue of understanding the power of different ordering strategies in influencing the performances of Second Order equilibria. An interesting question can be also that of trying to understand if the use of Second Order equilibria can lead faster convergence and/or convergence towards better states. In the paper we have only considered impatient prudent agents. A final open issue is certainly that of analyzing the other three possible definitions for rational agents.

Acknowledgements

The authors would like to thank an anonymous referee for pointing out interesting references on past works on farsighted equilibria.

References

[1] J. Aaftink, Far-sighted equilibria in 2 × 2, non-cooperative, repeated games, Theory and Decision 27 (1989) 175–192.
[2] E. Anshelevich, A. Dasgupta, J. Kleinberg, E. Tardos, T. Wexler, T. Roughgarden, The price of stability for network design with fair cost allocation, in: Proceedings of the 45th Annual IEEE Symposium on Foundations of Computer Science, FOCS'04, IEEE Computer Society, 2004, pp. 295–304.
[3] E. Anshelevich, A. Dasgupta, E. Tardos, T. Wexler, Near-optimal network design with selfish agents, in: Proceedings of the 35th Annual ACM Symposium on Theory of Computing, STOC'03, ACM Press, 2003, pp. 511–520.
[4] R.J. Aumann, Acceptable points in games of perfect information, Pacific Journal of Mathematics 10 (1960) 381–417.
[5] R.J. Aumann, Subjectivity and correlation in randomized strategies, Journal of Mathematical Economics 1 (1974) 67–96.
[6] M.J. Beckmann, On the Theory of Traffic Flow in Networks, in: Traffic Quart, vol. 21, 1967, pp. 109–116.
[7] M.J. Beckmann, C.B. McGuire, C.B. Winsten, Studies in the Economics of Transportation, Yale University Press, 1956.
[8] V. Bilò, On the packing of selfish items, in: Proceedings of the 20th IEEE International Parallel and Distributed Processing Symposium, IPDPS'06, IEEE Computer Society, 2006.
[9] V. Bilò, M. Flammini, L. Moscardelli, On nash equilibria in non-cooperative all-optical networks, in: Proceedings of the 22nd Annual Symposium on Theoretical Aspects of Computer Science, STACS'05, in: Lecture Notes in Computer Science, vol. 3404, Springer, 2005, pp. 448–459.
[10] V. Bilò, M. Flammini, G. Melideo, L. Moscardelli, On nash equilibria for multicast transmissions in Ad-hoc wireless networks, Wireless Networks 14 (2) (2008) 147–157.
[11] D. Braess, Uber ein Paradoxon der Verkehrsplanung, Unternehmensforschung 12 (1968) 258–268.
[12] S.J. Brams, D. Wittman, Non-myopic equilibria in 2 × 2 games, Conflict Management and Peace Science 6 (1981) 39–62.
[13] T. Bu, X. Deng, Q. Qi, Forward looking nash equilibrium for keyword auction, Information Processing Letters 105 (2008) 41–46.
[14] G. Christodoulou, E. Koutsoupias, A. Nanavati, Coordination Mechanisms, in: Proceedings of the 31st International Colloquium on Automata, Languages and Programming, ICALP'04, in: Lecture Notes in Computer Science, vol. 3142, Springer, 2004, pp. 345–357.
[15] A. Czumaj, B. Vocking, Tight bounds for worst-case equilibria, ACM Transactions on Algorithms 3 (1) (2007).
[16] A. Czumaj, P. Krysta, B. Vocking, Selfish traffic allocation for server farms, in: Proceedings of the 34th Annual ACM Symposium on Theory of Computing, STOC'02, ACM Press, 2002, pp. 287–296.
[17] S.C. Dafermos, F.T. Sparrow, The traffic assignment problem for a general network, Journal of Research of the National Bureau of Standards — B. Mathematical Sciences 73B (2) (1969) 91–118.
[18] E. Even-Dar, A. Kesselman, Y. Mansour, Convergence time to nash equilibria, ACM Transactions on Algorithms 3 (3) (2007).
[19] A. Fabrikant, A. Luthra, E. Maneva, C.H. Papadimitriou, S. Shenker, On a network creation game, in: Proceedings of the 22nd ACM Symposium on Principles of Distributed Computing, PODC'03, ACM Press, 2003, pp. 347–351.
[20] R. Feldmann, M. Gairing, T. L ucking, B. Monien, M. Rode, Nashification and the coordination ratio for a selfish routing game, in: Proceedings of the 30th International Colloquium on Automata, Languages and Programming, ICALP'03, in: Lecture Notes in Computer Science, vol. 2719, Springer, 2003, pp. 514–526.
[21] D. Fotakis, S. Kontogiannis, E. Koutsoupias, M. Mavronicolas, P. Spirakis, The structure and complexity of nash equilibria for a selfish routing game, in: Proceedings of the 29th International Colloquium on Automata, Languages and Programming, ICALP'02, in: Lecture Notes in Computer Science, vol. 2380, Springer, 2002, pp. 123–134.
[22] M. Gairing, T. Lucking, M. Mavronicolas, B. Monien, The price of anarchy for restricted parallel links, Parallel Processing Letters 16 (1) (2006) 117–132.
[23] M. Gairing, T. Lucking, M. Mavronicolas, B. Monien, The price of anarchy for polynomial social cost, Theoretical Computer Science 369 (1-3) (2006) 117–132.
[24] M. Gairing, T. Lucking, M. Mavronicolas, B. Monien, M. Rode, Nash equilibria in discrete routing games with convex latency functions, in: Proceedings of the 31st International Colloquium on Automata, Languages and Programming, in: Lecture Notes in Computer Science, vol. 3142, Springer, 2004, pp. 645–657.
[25] M. Gairing, T. Lucking, M. Mavronicolas, B. Monien, P. Spirakis, Structure and complexity of extreme nash equilibria, Theoretical Computer Science 343 (1-2) (2005) 133–157.
[26] M.X. Goemans, V.S. Mirrokni, A. Vetta, Sink equilibria and convergence, in: Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science, FOCS'05, IEEE Computer Society, 2005, pp. 142–154.
[27] J.C. Harsanyi, Games with incomplete information played by 'bayesian' players, Management Science 14 (1967) 159–182. 320-334, 486-502.
[28] M. Kandori, G. Mailath, R. Rob, Learning, mutation and long-run equilibria in games, Econometrica 61 (1993) 29–56.
[29] D.M. Kilgour, Equilibria for far-sighted players, Theory and Decision 16 (1984) 135–157.
[30] M. Kohlberg, J. Mertens, On the strategic stability of equilibria, Econometrica 54 (5) (1986) 1003–1037.
[31] E. Koutsoupias, M. Mavronicolas, P. Spirakis, Approximate equilibria and ball fusion, Theory of Computing Systems 36 (6) (2003) 683–693.
[32] E. Koutsoupias, C.H. Papadimitriou, Worst-case equilibria, in: Proceedings of the 16th International Symposium on Theoretical Aspects of Computer Science, in: Lecture Notes in Computer Science, vol. 1653, Springer, 1999, pp. 404–413.
[33] D. Kreps, R. Wilson, Sequential equilibria, Econometrica 50 (1982) 863–894.
[34] R.D. Luce, H. Raiffa, Games and Decisions: Introduction and Critical Survey, Wiley & Sons, New York, 1957.
[35] T. Lucking, M. Mavronicolas, B. Monien, M. Rode, A new model for selfish routing, in: Proceedings of the 21st Annual Symposium on Theoretical Aspects of Computer Science, STACS'04, in: Lecture Notes in Computer Science, vol. 2996, Springer, 2004, pp. 547–558.
[36] T. Lucking, M. Mavronicolas, B. Monien, M. Rode, P. Spirakis, I. Vrto, Which is the worst-case nash equilibrium? in: Proceedings of the 28th International Symposium of Mathematical Foundations of Computer Science, MFCS'03, in: Lecture Notes in Computer Science, vol. 2747, Springer, 2003, pp. 551–561.
[37] M. Mavronicolas, P. Spirakis, The price of selfish routing, Algorithmica 48 (1) (2007) 91–126.
[38] N. Nakanishi, Purely noncooperative farsighted stable set in an n-player prisoners dilemma, International Journal of Game Theory (in press).
[39] J. Nash, Equilibrium points in n-person games, in: Proceedings of the National Academy of Sciences, vol. 36, 1950, pp. 48–49.
[40] J. Nash, Non-cooperative games, Annals of Mathematics 54 (2) (1951) 286–295.
[41] C.H. Papadimitriou, Algorithms, games, and the internet, in: Proceedings of the 33rd Annual ACM Symposium on Theory of Computing, ACM Press, 2001, pp. 749–753.
[42] R.W. Rosenthal, A class of games possessing pure-strategy nash equilibria, International Journal of Game Theory 2 (1973) 65–67.
[43] T. Roughgarden, The price of anarchy is independent of the network topology, Journal of Computer and System Sciences 67 (2) (2003) 341–364.
[44] T. Roughgarden, Selfish routing. Ph.D. Thesis, Department of Computer Science, Cornell University, May 2002.
[45] T. Roughgarden, E. Tardos, How bad is selfish routing? Journal of ACM 49 (2002) 236–259.
[46] R. Selten, Spieltheoretische behandlung eines oligopolmodells mit nachfragentragheit, Zeitschrift fur die gesamte Staatswissenschaft 12 (1965) 201–324.
[47] R. Selten, Reexamination of the perfectness concept for equilibrium points in extensive games, International Journal of Game Theory 4 (1975) 25–55.
[48] J. von Neumann, O. Morgenstern, Theory of games and economic behavior, third ed, Princeton University Press, 1953.
[49] J.G. Wardrop, Some theoretical aspects of road traffic research, in: Proceedings of the Institute of Civil Engineers, Pt. II, Vol. 1, 1956, pp. 325–378.